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One modification to the Yakhot-Orszag calculation in the renormalization-group theory of turbulence

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In the calculation of the Kolmogrov constant, by applying the renormalization-group method to turbulence, Yakhot and Orszag choose $\epsilon=0$ and 4 at different points in the same calculation [J. Sci. Comput. **1**, 3 (1986); Phys. Rev. Lett. **57**, 1722 (1986); Nucl. Phys. B **2**, 417 (1987)]. Modifying their calculation for the renormalized viscosity, the Kolmogrov constant that is in agreement with experimental data can be evaluated only by using $\epsilon=4$. Results that make the theory consistent are obtained.

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Yakhot and Orszag [1–3] have applied the renormalization-group (RNG) method to practical turbulence problems. Eliminating all modes with the wave number $q>k$, they have calculated the renormalized viscosity

$$\nu(k)=\left[\frac{3}{\epsilon}A_dD_0\right]^{1/3}k^{-\epsilon/3}$$

and the energy spectrum

$$E(k)=\frac{D_0\dfrac{S_d}{(2\pi)^d}}{\left[\dfrac{3}{\epsilon}A_dD_0\right]^{1/3}}k^{1-(2/3)\epsilon},$$

where

$$A_d=\tilde{A}_d\frac{S_d}{(2\pi)^d},\quad \tilde{A}_d=\frac{1}{2}\frac{d^2-d-\epsilon}{d(d+2)}.$$

Choosing $\epsilon=4$, the energy spectrum satisfying the Kolmogrov $-5/3$ power law can be obtained. However, $\tilde{A}_d$ is evaluated at $\epsilon=0$ as $\tilde{A}_3(\epsilon=0)=0.2$ for the calculation of the Kolmogrov constant in agreement with the experimental data [1–3]. For the above reason, the renormalization-group theory of turbulence has raised doubts [4].

In this paper, the Yakhot-Orszag calculation for the renormalized viscosity is modified. We obtain the results that $\tilde{A}_d$ is not related to ϵ and equals 0.2 (for $d=3$). Thus the Kolmogrov constant that is in agreement with experimental data can be evaluated only by using $\epsilon=4$ consistently. The turbulent flow in the inertial range can be described by the Navier-Stokes equations with a random force:

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla P+\nu_0\nabla^2\mathbf{v}, \tag{1}$$

where the Gaussian random force $\mathbf{f}$ is specified by the two-point correlation,

$$\langle f_i(\hat{k})f_j(\hat{k}')\rangle=2D_0k^{-y}(2\pi)^{d+1}P_{ij}(\mathbf{k})\delta(\hat{k}+\hat{k}'), \tag{2}$$

where $\hat{k}=(\mathbf{k},\omega)$. Choosing an ultraviolet cutoff $\Lambda=O(k_d)$, the Fourier decomposition is utilized to transform the problems in physical space to that in spectral space. The velocity $\mathbf{v}$ is divided into two components $\mathbf{v}^>$ and $\mathbf{v}^<$, with the wave number satisfying $\Lambda e^{-r}<q<\Lambda$ and $q<\Lambda e^{-r}$. Eliminating the modes in the internal $\Lambda e^{-r}<q<\Lambda$, the equations for $\mathbf{v}^<$ are given as follows [1–3]:

$$(-i\omega+\nu_0k^2)v_l^<(\hat{k})=f_1(\hat{k})-\frac{i\lambda_0}{2}P_{lmn}(\mathbf{k})\int v_m^<(\hat{\mathbf{q}})v_n^<(\hat{k}-\hat{\mathbf{q}})\frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}}+R_1, \tag{3}$$

in which

$$R_1=-\lambda_0^2\frac{D_0}{(2\pi)^d\nu_0}P_{lmn}(\mathbf{k})\int_{\Lambda e^{-r}<q<\Lambda}\frac{P_{n\mu\rho}(\mathbf{k}-\mathbf{q})P_{m\mu}(\mathbf{q})q^{-y-2}}{-i\omega+\nu_0q^2+\nu_0|\mathbf{k}-\mathbf{q}|^2}d\mathbf{q}v_\rho^<(\hat{k}). \tag{4}$$

Changing the integration variable by replacing $\mathbf{q}\to\mathbf{q}+\frac{1}{2}\mathbf{k}$, gives

$$R_1=-\lambda_0^2\frac{D_0}{(2\pi)^d\nu_0}P_{lmn}(\mathbf{k})\int_{\Lambda e^{-r}<|\mathbf{q}-(1/2)\mathbf{k}|<\Lambda}\frac{P_{n\mu\rho}(\frac{1}{2}\mathbf{k}-\mathbf{q})P_{m\mu}(\mathbf{q}+\frac{1}{2}\mathbf{k})|\mathbf{q}+\frac{1}{2}\mathbf{k}|^{-y-2}}{-i\omega+2\nu_0q^2+\frac{1}{2}\nu_0k^2}d\mathbf{q}v_\rho^<(\hat{k}). \tag{5}$$

For evaluating the integration on the right-hand side of (5) easily, Yakhot and Orszag [1] replace the integral region $\Lambda e^{-r}<|\mathbf{q}-\frac{1}{2}\mathbf{k}|<\Lambda$ with $\Lambda e^{-r}<q<\Lambda$. In the limit $k\to0$, $\omega\to0$, the result is

$$R_1=-A_d\frac{\lambda_0^2D_0}{\nu_0^2\Lambda^\epsilon}\frac{e^{\epsilon r}-1}{\epsilon}k^2v_1^<(\hat{k}). \tag{6}$$

From (6), the formulas for the renormalized viscosity and the energy spectrum can be obtained [1–3].

But the replacement of the integral region $\Lambda e^{-r} < |\mathbf{q}-\frac{1}{2}\mathbf{k}| < \Lambda$ with $\Lambda e^{-r} < q < \Lambda$ is not appropriate, since its consequence is that the Kolmogrov constant, in agreement with the experimental data, should be evaluated using $\epsilon=0$ and 4 at different points in the same calculation. We modify the Yakhot-Orszag calculation for the integration on the right-hand side of (4). In the limit $k \to 0$, $\omega \to 0$, by expanding the integrand on the right-hand side of (4) into the Taylor's series and neglecting terms that are $O(k^2)$ in this integrand, we have

$$R_1 = -\lambda_0^2 \frac{D_0}{(2\pi)^d \nu_0} P_{lmn}(\mathbf{k}) \int_{\Lambda e^{-r} < q < \Lambda} \left\{ \frac{\frac{d}{dp_\alpha}[P_{n\mu\rho}(-\mathbf{p})](-k_\alpha)}{-i\omega + \nu_0 q^2 + \nu_0 p^2} \right\} \Bigg|_{\mathbf{p}=\mathbf{q}} P_{m\mu}(\mathbf{q}) q^{-y-2} v_\rho^<(\hat{k}) d\mathbf{q} - \lambda_0^2 \frac{D_0}{(2\pi)^d \nu_0} P_{lmn}(\mathbf{k})$$
$$\times \int_{\Lambda e^{-r} < q < \Lambda} \left\{ \frac{d}{dp_\alpha} \left[\frac{1}{-i\omega + \nu_0 q^2 + \nu_0 p^2} \right] P_{n\mu\rho}(-\mathbf{p}) \right\} \Bigg|_{\mathbf{p}=\mathbf{q}}$$
$$\times (-k_\alpha) P_{m\mu}(\mathbf{q}) q^{-y-2} v_\rho^<(\hat{k}) d\mathbf{q} \ . \tag{7}$$

Noticing the continuous condition $k_\rho v_\rho^<(\hat{k})=0$ and using the standard identities,

$$\int q_\alpha q_\beta d\mathbf{q} = \frac{S_d}{d} \delta_{\alpha\beta} \int q^{d+1} dq \ , \tag{8}$$

$$\int q_\alpha q_\beta q_\gamma q_\delta d\mathbf{q} = \frac{S_d}{d(d+2)} (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta}) \times \int q^{d+3} dq \ , \tag{9}$$

the results for the first term R_{11} and the second term R_{12} on the right-hand side of (7) are given as follows:

$$R_{11} = -\frac{\lambda_0^2 D_0}{\nu_0^2 \Lambda^\epsilon} \frac{S_d}{(2\pi)^d} \frac{d^2-d-2}{2d(d+2)} \frac{e^{\epsilon r}-1}{\epsilon} k^2 v_1^<(\hat{k}) \ , \tag{10}$$

$$R_{12} = -\frac{\lambda_0^2 D_0}{\nu_0^2 \Lambda^\epsilon} \frac{S_d}{(2\pi)^d} \frac{1}{d(d+2)} \frac{e^{\epsilon r}-1}{\epsilon} k^2 v_1^<(\hat{k}) \ . \tag{11}$$

From (10) and (11), we have

$$R_1 = -A_d \frac{\lambda_0^2 D_0}{\nu_0^2 \Lambda^\epsilon} \frac{e^{\epsilon r}-1}{\epsilon} k^2 v_1^<(\hat{k}) \ , \tag{12}$$

where

$$A_d = \tilde{A}_d \frac{S_d}{(2\pi)^d}, \quad \tilde{A}_d = \frac{1}{2} \frac{d^2-d}{d(d+2)} \ . \tag{13}$$

It is easy to show that our calculation for $\tilde{A}_d$ in (13) is equal to the Yakhot-Orszag calculation for $\tilde{A}_d(\epsilon=0)$. Thus we propose that the energy spectrum takes the Kolmogrov form, and that the Kolmogrov constant, in agreement with the experimental data, be calculated by only using $\epsilon=4$, consistently modifying the Yakhot-Orszag calculation for the renormalized viscosity.

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